Exactly soluble noisy traveling-wave equation appearing in the problem of directed polymers in a random medium

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We calculate exactly the velocity and diffusion constant of a microscopic stochastic model of N evolving particles which can be described by a noisy traveling-wave equation with a noise of order $N^{-1/2}$. Our model can be viewed as the infinite range limit of a directed polymer in random medium with N sites in the transverse direction. Despite some peculiarities of the traveling-wave equations in the absence of noise, our exact solution allows us to test the validity of a simple cutoff approximation and to show that, in the weak noise limit, the position of the front can be completely described by the effect of the noise on the first particle.

Traveling-wave equations such as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [1–3],

$$\partial_t u = \partial_x^2 u + u - u^2, \tag{1}$$

describe moving fronts [4] in a large number of problems in biology, chemistry, and genetics. In physics, it appears in nonequilibrium statistical mechanics and in the theory of disordered systems [5,6]. In typical cases, $u_t(x)$ represents the concentration at position x and time t of some chemical species or of individuals carrying a certain gene. It is well established [3,4,7] that equations of the FKPP type have a one parameter family of traveling-wave solutions with $u(-\infty) = 1$ and $u(\infty) = 0$ parametrized by their velocities v, and that the decay of the initial condition determines the velocity of the front. For localized initial conditions, the velocity is the minimal velocity allowed $v_{\min} = 2$.

When deriving traveling-wave equations such as Eq. (1) from a given microscopic stochastic models [8–13], one usually gets a noisy version of Eq. (1):

$$\partial_t u = \partial_x^2 u + u - u^2 + \sqrt{\epsilon}\, g(u)\xi(x,t), \tag{2}$$

where the additional term is proportional to a function $g(u)$ of the concentration and to a Gaussian noise $\xi(x,t)$ white in time and possibly correlated in space. The amplitude ϵ represents the ratio between the microscopic and the macroscopic scales. Internal fluctuations due to the finite number of interacting particles give [11,12,14,15] $g(u) \propto \sqrt{u(1-u)}$. Although it is now established [11] that the presence of the noise term is sufficient [8–10,16] to select a single velocity v_ϵ, and that this selected velocity v_ϵ tends slowly [9,10] in the limit $\epsilon \to 0$ to the minimal velocity $v_{\min}$ allowed by Eq. (1), it is still a theoretical challenge [4,10,12,17,18] to predict how $v_{\min} - v_\epsilon$ vanishes with ϵ or how specific properties of the noisy equation such as the diffusion constant D_ϵ of the front position behave in the small noise limit.

Other types of noise have also been considered; for instance, taking $g(u) \propto u$ in Eq. (2) would represent the fluctuations of a control parameter due to some external noise [19,20]. We will not discuss these other types of noise in the present work.

Here, we consider a microscopic model which can be viewed as the problem of directed polymers in a random medium [21] where we take the infinite range limit in the transverse direction. As explained below, this problem leads to a noisy traveling-wave equation and, for one specific choice of the disorder in the directed polymer problem, we can solve the microscopic dynamics and calculate exactly the velocity, diffusion constant, and all the higher cumulants of the position of the front. This exact solution allows us to test several approximation schemes used recently to attack the general problem of noisy fronts.

The microscopic model we consider here is defined as follows: at each time step t we have N particles on the real axis at positions $x_1(t), \ldots, x_N(t)$. Given these positions at time t, the new positions at time $t+1$ are obtained by

$$x_i(t+1) = \max_{1 \le j \le N} [x_j(t) + s_{i,j}(t)], \tag{3}$$

where the $s_{i,j}(t)$ are independent random numbers generated according to a given distribution $\rho(s)$. One can think of $-x_i(t)$ as being the ground state energy of a directed polymer of length t ending at position i in the "transverse" direction. Then Eq. (3) describes the infinite range case in the transverse direction as the directed polymer may jump at any step t from any site j to any site i and gain a bond energy $-s_{i,j}(t)$. It is easy to show that the points remain grouped and we want to know the velocity v_{exact} and the diffusion D_{exact} of this cloud of points (or, equivalently, of its center of mass).

One can associate a traveling-wave equation to Eq. (3) by considering the proportion $u_t(x)$ of particles on the right of x:

$$u_t(x) = \frac{1}{N} \sum_{1 \le i \le N} \theta(x_i(t) - x), \tag{4}$$

where $\theta(z)$ is the Heaviside function. Clearly, $u_t(x)$ is a decreasing function with $u_t(-\infty) = 1$ and $u_t(\infty) = 0$, so that $u_t(x)$ has the shape of a front.

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For a given front configuration $u_t(x)$, the random variables $\{x_i(t+1)\}$ are uncorrelated and the probability to be on the right of x is given by

$$\text{Prob}[x_i(t+1) > x] = 1 - \prod_{1 \leq j \leq N} \mu(x - x_j(t)), \quad (5)$$

where $\mu(s)$ is the probability that $s_{ij}(t) < s$:

$$\mu(s) = \int_{-\infty}^{s} \rho(s')ds'. \quad (6)$$

For a given $u_t(x)$, the probability (5) is simply the front $\langle u_{t+1}(x) \rangle$ at time $t+1$ averaged *over one time step*. One could rewrite Eq. (5) using only the front variable $u_t(x)$:

$$\langle u_{t+1}(x) \rangle = 1 - \exp\left[-N \int u_t(y) \frac{\mu'(x-y)}{\mu(x-y)} dy \right], \quad (7)$$

but one needs to be careful when μ vanishes and use the following prescription: When $\mu(x-y)=0$, then $u_t(y)\mu'(x-y)/\mu(x-y)$ should be set to 0 if $u_t(y)=0$ and to $+\infty$ if $u_t(y) \neq 0$.

The random positions $x_i(t+1)$ can be generated by solving $\langle u_{t+1}(x_i) \rangle = z_i$, where $z_1, \ldots, z_N$ are N independent random numbers uniformly chosen between 0 and 1 and the fluctuating front $u_{t+1}(x)$ is finally given by

$$u_{t+1}(x) = \frac{1}{N} \sum_{1 \leq i \leq N} \theta(\langle u_{t+1}(x) \rangle - z_i). \quad (8)$$

From Eq. (8), one can calculate how the fluctuations of $u_{t+1}(x)$ are correlated. Writing

$$u_{t+1}(x) = \langle u_{t+1}(x) \rangle + \frac{1}{\sqrt{N}} \eta_{t+1}(x), \quad (9)$$

one finds $\langle \eta_{t+1}(x) \rangle = 0$ and for $x \leq y$

$$\langle \eta_{t+1}(x) \eta_{t+1}(y) \rangle = [1 - \langle u_{t+1}(x) \rangle] \langle u_{t+1}(y) \rangle. \quad (10)$$

Writing $\eta(x) = \xi(x)\sqrt{\langle \eta^2(x) \rangle}$, Eq. (9) becomes very similar to Eq. (2) with $g(u) = \sqrt{u(1-u)}$. So far, Eqs. (7) and (8) and their consequence Eqs. (9) and (10) are exact, for arbitrary N. From Eq. (8) one can also calculate higher correlations of $\eta_{t+1}(x)$ and show that, for large N, the η become Gaussian. For example, one can check that, up to terms of order $1/N$,

$$\langle \eta_1 \eta_2 \eta_3 \eta_4 \rangle = \langle \eta_1 \eta_2 \rangle \langle \eta_3 \eta_4 \rangle + \langle \eta_1 \eta_3 \rangle \langle \eta_2 \eta_4 \rangle + \langle \eta_1 \eta_4 \rangle \langle \eta_2 \eta_3 \rangle$$

[we used the simplified notation $\eta_i \equiv \eta_{t+1}(x_i)$]. We should notice that the Gaussian character of the $\eta_{t+1}(x)$ is a property valid only for large N and in regions where

$$N(1 - \langle u_{t+1}(x) \rangle) \gg 1 \quad \text{and} \quad N\langle u_{t+1}(x) \rangle \gg 1. \quad (11)$$

From now on, we will limit our discussion to the case where $\rho(s)$ is a Gumbel distribution:

$$\rho(s) = \exp(-s - e^{-s}). \quad (12)$$

In that case, the full analysis of the problem becomes easy and we can calculate exactly the statistical properties of the front in the large N limit. From Eqs. (6) and (12), one has $\mu'(s)/\mu(s) = e^{-s}$ and Eq. (7) becomes

$$\langle u_{t+1}(x) \rangle = 1 - \exp[-B_t e^{-x}], \quad (13)$$

where B_t is defined by

$$B_t = N \int e^x u_t(x)dx = \sum_{1 \leq i \leq N} e^{x_i(t)}. \quad (14)$$

This definition and Eq. (13) imply that, given B_t,

$$\langle B_{t+1} \rangle = \infty, \quad (15)$$

which means that the distribution of the random variable B_{t+1} decays slowly.

The main advantage of the Gumbel distribution (12) is that the maximum of several Gumbel variables is itself distributed according to a Gumbel distribution. Therefore, for fixed $x_j(t)$ in Eq. (3), having the $s_{i,j}(t)$ distributed according to the Gumbel distribution (12) implies that $x_{i+1}(t)$ is itself a Gumbel variable.

The Gumbel distribution is simple because B_t in Eq. (14) is the only information needed to construct the front $u_{t+1}(x)$ at time $t+1$. If one defines the position X_t of the front at time t by $X_t = \ln B_t$, then the displacements

$$\Delta X_t = X_{t+1} - X_t = \ln B_{t+1} - \ln B_t \quad (16)$$

are uncorrelated random variables given by

$$\Delta X_t = \ln\left[\frac{1}{y_1(t+1)} + \cdots + \frac{1}{y_N(t+1)} \right], \quad (17)$$

where the $y_i(t+1) = B_t \exp[-x_i(t+1)]$ are independent and, using Eqs. (5), (7), and (13), distributed according to

$$p(y) = e^{-y}\theta(y). \quad (18)$$

As the ΔX_t are independent, the cumulants of the position X_t are simply t times those of ΔX_t, which can be calculated from its generating function. The identity

$$\langle e^{-\delta \Delta X_t} \rangle = \frac{1}{\Gamma(\delta)} \int_0^{\infty} du \, u^{\delta-1} \langle \exp(-u e^{\Delta X_t}) \rangle, \quad (19)$$

and the fact that the $y_i(t)$ are independently distributed according to Eq. (18), lead to the following exact expression:

$$\langle e^{-\delta \Delta X_t} \rangle = \frac{1}{\Gamma(\delta)} \int_0^{\infty} du \, u^{\delta-1} \left(\int_0^{\infty} e^{-y-(u/y)} dy \right)^N. \quad (20)$$

For large N, the integral over u is dominated by the neighborhood of $u=0$ where

$$\int_0^{\infty} e^{-y-(u/y)} dy = 1 + u \ln u + (2\gamma_E - 1)u + O(u^2 \ln u), \quad (21)$$

where $\gamma_E = -\Gamma'(1) = -\int_0^{+\infty} du \, e^{-u} \ln u$ is the Euler gamma constant. Any term of higher order in u would give a correction $1/N$ to the final result. Using Eq. (21) into Eq. (20), one obtains, for large N,

$$\ln\langle e^{-\delta\Delta X_t}\rangle = -\delta(L + \ln L)$$
$$-\frac{\delta}{L}\left[\ln L + 1 - 2\gamma_E - \frac{\Gamma'(1+\delta)}{\Gamma(1+\delta)}\right] + o\left(\frac{1}{L}\right), \tag{22}$$

where $L \equiv \ln N$. This leads to the following exact expressions for the velocity and of the diffusion constant of the front described by Eqs. (7)–(10) when $\rho(s)$ is given by Eq. (12):

$$v_{\text{exact}} = \lim_{t\to\infty}\frac{\langle X_t\rangle}{t} = L + \ln L + \frac{\ln L}{L} + \frac{1-\gamma_E}{L} + o\left(\frac{1}{L}\right),$$

$$D_{\text{exact}} = \lim_{t\to\infty}\frac{\langle X_t^2\rangle - \langle X_t\rangle^2}{t} = \frac{\pi^2}{3L} + o\left(\frac{1}{L}\right). \tag{23}$$

Expanding Eq. (22) in powers of δ gives also all the higher cumulants of the position of the front.

Armed with these exact results, one can test the quality of various approximation schemes [10,22].

A first approximation is to neglect the noise and to write a deterministic traveling-wave equation for the front. However, if one replaces $\langle u_{t+1}(x)\rangle$ by $u_{t+1}(x)$ in Eq. (13), one obtains a meaningless front equation: starting with $u_0(x) = \theta(-x)$, one finds $u_2(x) = 1$ for all x. Equation (13) is meaningful only in the presence of noise.

Another way of removing the noise is to assume that for each position $x_i(t+1)$ in Eq. (5), all the $x_j(t)$ are uncorrelated random variables chosen independently for each i according to the distribution $-\partial u_t(x)/\partial x$. This leads to the following deterministic equation of a front propagating into an unstable state:

$$u_{t+1}(x) = 1 - \left[1 - \int dy\, u_t(y)\rho(x-y)\right]^N. \tag{24}$$

This equation is like Eq. (1), a traveling wave equation, and its velocity can be obtained using the usual method [4]: looking for solutions of the form $u_t(x) = \exp\{-\gamma[x - v(\gamma)t]\}$ when $u_t(x) \ll 1$, one obtains a function $v(\gamma)$. For initial conditions which decay fast enough, the velocity is $v_{\text{meanfield}} = \min_\gamma v(\gamma)$. When $\rho(s)$ is the Gumbel distribution, one gets $v(\gamma) = [\ln N + \ln\Gamma(1-\gamma)]/\gamma$ and, for large N, the minimal velocity is $v_{\text{meanfield}} = v_{\text{exact}} + 1 + o(1/L)$. One could try to improve this result by using the cutoff approximation [10,17,22] in Eq. (24), but as $v(\gamma)$ depends on N, it is not clear that the formula $\Delta v = -\pi^2\gamma^2 v''(\gamma)/(2\ln^2 N)$ of Ref. [10] can be applied. Trying to apply it anyway, one obtains the velocity $v_{\text{meanfield}} - \pi^2/2$ which is not closer to v_{exact} than $v_{\text{meanfield}}$.

The cutoff approximation can also be applied directly on the evolution equations (13) and (14) by setting $u_{t+1}(x) = \langle u_{t+1}(x)\rangle$ whenever $\langle u_{t+1}(x)\rangle > \lambda/N$ and $u_{t+1}(x) = 0$ otherwise, where λ is an arbitrary number of order 1. One can then write a closed expression for the evolution of B_t:

$$B_{t+1} = N\int_{-\infty}^{A_{t+1}} e^x(1 - e^{-B_t e^{-x}})dx, \tag{25}$$

where the position A_{t+1} of the cutoff is given by

$$\frac{\lambda}{N} = \langle u_{t+1}(A_{t+1})\rangle = 1 - e^{-B_t e^{-A_{t+1}}}. \tag{26}$$

By eliminating A_{t+1}, one gets

$$B_{t+1} = B_t N\int_{\ln(N/N-\lambda)}^{\infty}\frac{1-e^{-u}}{u^2}du. \tag{27}$$

Using $v_{\text{cutoff}} = \ln(B_{t+1}/B_t)$, we get, for large $L \equiv \ln N$,

$$v_{\text{cutoff}} = L + \ln L + \frac{1 - \gamma_E - \ln\lambda}{L} + o\left(\frac{1}{L}\right). \tag{28}$$

Comparing Eqs. (28) and (23), we see that v_{cutoff} gives correctly the leading orders in L with a discrepancy $(\ln L)/L$, which is only slightly above what the cutoff approximation may predict anyway, as there is no reason to choose any particular value of λ, and which is much better than the mean-field velocity obtained from Eq. (24).

So far, all the approximations replaced the noisy dynamics by a deterministic equation. Thus they gave no prediction for the diffusion constant. We will now examine approximations in which the system remains noisy.

A first possibility is to consider that the evolution is given by Eqs. (9), (10), (13), and (14) where the noise term in Eq. (9) is exactly Gaussian, even outside the validity range (11). In this case, B_{t+1} is Gaussian but with $\langle B_{t+1}\rangle = \infty$ [see Eq. (15)]. The noisy front equation (9) becomes meaningless after a single time step. So, one cannot ignore that the noise is not Gaussian near the rightmost particle.

The next approximation one can try is to put a cutoff into the system as in Eq. (26), and to take into account the effect of all the fluctuations on the left of this cutoff. In other words, at each time step, we determine A_{t+1} by Eq. (26), we set $u_{t+1}(x) = 0$ for $x > A_{t+1}$ and we use Eq. (9) for $x < A_{t+1}$ with a *Gaussian* noise $\eta_{t+1}(x)$ correlated as in Eq. (10). This leads to a Gaussian B_{t+1} characterized by $\langle B_{t+1}\rangle$ given by the right hand side of Eq. (25) and (27) and

$$\langle B_{t+1}^2\rangle - \langle B_{t+1}\rangle^2 = 2N\int_{-\infty}^{A_{t+1}} dy(1 - e^{-B_t e^{-y}})e^y$$
$$\times \int_{-\infty}^{y} dx\, e^x e^{-B_t e^{-x}}. \tag{29}$$

For large N, one easily gets $\langle B_{t+1}\rangle \simeq NB_t \ln N$ and $\langle B_{t+1}^2\rangle - \langle B_{t+1}\rangle^2 \simeq 2N^2 B_t^2/\lambda$, giving for the diffusion constant of the front position X_t

$$D \simeq \frac{\langle B_{t+1}^2\rangle - \langle B_{t+1}\rangle^2}{\langle B_{t+1}\rangle^2} \simeq \frac{2}{\lambda L^2}. \tag{30}$$

Compared to the exact result (23), this has the wrong L dependence and also depends on the precise value of the cutoff λ. The velocity $v_{\text{cutoff}} - D/2$ obtained in this case is also not a better approximation than the cutoff velocity (28). We conclude that the effect of the noise at the left of the cutoff in the present model is too small to explain the value of the exact diffusion constant (23) and can therefore be neglected.

A last approximation is to keep as only source of noise the stochastic position x_{max} of the rightmost particle [22]: we

choose x_{max} using the distribution of the exact front dynamics and, on the left of x_{max}, we use for $u_{t+1}(x)$ the average value of the front *given that the rightmost particle is on x_{max}*. In other words:

$$u_{t+1}(x) = \begin{cases} 0 & \text{if } x > x_{max}, \\ \dfrac{1}{N} + \dfrac{N-1}{N}\langle u_{t+1}(x)\rangle & \text{if } x < x_{max}. \end{cases} \quad (31)$$

From Eq. (13), one gets

$$x_{max} = \ln\frac{NB_t}{q} \quad \text{with Prob}(q) = e^{-q}\theta(q), \quad (32)$$

and from Eq. (14),

$$\frac{B_{t+1}}{B_t} = \frac{N}{q} + (N-1)\int_{q/N}^{+\infty}\frac{1-e^{-u}}{u^2}du. \quad (33)$$

Notice that, in that approximation, $\langle B_{t+1}\rangle = \infty$ due to the contribution of small q or, equivalently, of large x_{max}. For q small compared to N, one obtains

$$\frac{B_{t+1}}{B_t} = N\left[\frac{1}{q} + \ln N + 1 - \gamma_E - \ln q + o(1)\right]. \quad (34)$$

Using the definition (16) of the displacement ΔX_t and the identity (19), one can compute from Eq. (34) the generating function $\langle\exp(-\delta\Delta X_t)\rangle$. Up to order $1/\ln N$, one finds nearly the same result as in the exact solution: one just needs to replace $1 - 2\gamma_E$ in Eq. (22), by $2 - 2\gamma_E$. This means that, up to order $1/\ln N$, the velocity of the front in this approximation is shifted by the small amount $1/\ln N$, and all the other cumulants are the same as in the exact model.

In this work, we have shown how the problem of directed polymers with N sites in $1+\infty$ dimension can be reduced to a noisy traveling-wave equation (7) and (8). For one special choice of the bond disorder (12), we could calculate exactly the velocity and diffusion constant (23) of the front, and even

all the cumulants (22). The reason which makes this case soluble is that, at each time step, the only information one needs to keep about the past is a single variable B_t given by Eq. (14). This is similar to what was observed recently for shocks in exclusion processes [23,24] where, in certain cases, one can decouple the evolutions of the position of the shock and of its shape. Comparing several approximation schemes has shown that, in the present case, the cutoff approximation [10,12,17,25] gives a good estimate of the front velocity, and that the full large N fluctuations of the front can be obtained by considering the effect of the noise on the rightmost particle. The predominance of this rightmost particle might be related to some difficulties noticed in a previously studied growth model [26] where only the first cumulants of the heights were considered.

The front described by Eqs. (7) and (8) is peculiar because N appears both in the noise term and the traveling-wave equation itself and because neglecting the noise in Eq. (7) leads to an ill defined traveling-wave equation. In the mean field approximation, one obtains a FKPP-like front equation (24) which still depends on N and its velocity diverges like $\ln N$. These peculiarities make the problem considered here rather different from usual traveling-wave equations and does not allow us to use the present exact solution to check the validity of the $\ln^{-2}N$ shift of the velocity and of the $\ln^{-3}N$ dependence of the diffusion constant which have been suggested by a number of numerical calculations [22,27]. The approximations (cutoff or noise limited to the rightmost particle) successfully tested here should, however, be helpful to describe more standard front equations.

Of course, from both the points of view of the theory of disordered systems and of the theory of noisy traveling-wave equations, it would be interesting to attack the case of a general distribution $\rho(s)$. A starting point could be to try to make a perturbation theory with the Gumbel distribution as a zeroth order approximation. No need to say that obtaining the $\ln^{-2}N$ correction to the velocity would require a delicate resummation of this perturbation theory.

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